TRU Math Competition Practice Problem 2

Deadline: October 15, 2018

If n is a positive integer, prove that

$$\int_0^1 (\ln x)^n dx = (-1)^n n!.$$

Solution: Start with integration by parts $u = (\ln x)^n, dv = dx$ and get $du = \frac{n(\ln x)^{n-1}}{x}, v = x$. So

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$

Apply the integration by parts a few more times to get

$$\begin{aligned}\int (\ln x)^n dx &= x(\ln x)^n - nx(\ln x)^{n-1} + n(n-1)x(\ln x)^{n-2} \\ &+ \cdots + (-1)^{n-1} n(n-1)(n-2)\cdot \ldots \cdot 2x(\ln x) \\ &+ (-1)^n n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1x.\end{aligned}$$

Using L'Hospital's Rule repeatedly, we can see that $\lim_{t\to 0^+} x(\ln x)^{n-k} = 0$, for every $k = 0, \cdots, n-1$. Evaluating the following improper integral

$$\begin{aligned}\int_t^1 (\ln x)^n dx &= x(\ln x)^n]_t^1 - nx(\ln x)^{n-1}]_t^1 + n(n-1)x(\ln x)^{n-2}]_t^1 \\ &+ \cdots + (-1)^{n-1} n(n-1)(n-2)\cdot \ldots \cdot 2x(\ln x)]_t^1 \\ &+ (-1)^n n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1x]_t^1\end{aligned}$$

and sending t to zero finishes the proof.