Practice Problem 3
Deadline: October 8, 2018

Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

Solution: If at least one of $f(a), f'(a), f''(a)$, or $f'''(a)$ vanishes at some point a, we are done. Otherwise, by the Intermediate Value Theorem each of $f(x), f'(x), f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line, and we only need to show that their product is positive for a single value of x. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume that $f''(x) > 0$; by replacing $f(x)$ by $f(-x)$ if necessary, we may assume that $f'''(x) > 0$. Notice that these substitutions do not change the sign of $f(x) \cdot f'(x) \cdot f''(x) \cdot f'''(x)$.

Now $f'''(x) > 0$ implies that $f''(x)$ is increasing. Thus for $a > 0$,

$$f'(a) = f'(0) + \int_0^a f''(t)dt \geq f'(0) + af''(0).$$

In particular, $f'(a) > 0$ for large a. Similarly, since $f''(x) > 0$, $f(a)$ is positive for large a. Therefore, $f(x) \cdot f'(x) \cdot f''(x) \cdot f'''(x) > 0$ for sufficiently large x.