

TRU Math Competition Practice Problem 3

Deadline: October 22, 2018

Let a be a positive real number that is not an integer and let

$$n = \lfloor \frac{1}{a - \lfloor a \rfloor} \rfloor.$$

Prove that $\lfloor (n+1)a \rfloor - 1$ is divisible by $n+1$. [Hint: $\lfloor x \rfloor$ denotes the largest integer so that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. (University of Toronto, Undergraduate Mathematics Competition)]

Solution: Since

$$n \leq \frac{1}{a - \lfloor a \rfloor} < n + 1,$$

and hence

$$n(a - \lfloor a \rfloor) \leq 1 < (n + 1)(a - \lfloor a \rfloor),$$

we have

$$1 + (n + 1)\lfloor a \rfloor < (n + 1)a \leq 1 + n\lfloor a \rfloor + a < 2 + (n + 1)\lfloor a \rfloor$$

and therefore $\lfloor (n + 1)a \rfloor = 1 + (n + 1)\lfloor a \rfloor$.