TRU Math Competition Practice Problem 4

Deadline: October 29, 2018

Let $f : [0,1] \to [0,1]$ be a surjective function such that

$$|f(x) - f(y)| \geq |x - y|,$$

for all $x, y \in [0,1]$. Show that f is continuous.

Solution: We first show that f is injective: Let $f(x_1) = f(x_2)$. Then $0 = |f(x_1) - f(x_2)| \geq |x_1 - x_2|$ implies that $x_1 = x_2$.

So, f is a bijection and hence it has an inverse. We show that f^{-1} is continuous at every point in $[0,1]$: For an arbitrary $\varepsilon > 0$, we define $\delta := \varepsilon$ and hence if $|x - y| < \delta$, then

$$|f^{-1}(x) - f^{-1}(y)| \leq |f(f^{-1}(x)) - f(f^{-1}(y))| = |x - y| < \delta = \varepsilon.$$

So, f^{-1} is continuous. Since $[0,1]$ is compact, f^{-1} is a closed map and hence the image of any closed set in $[0,1]$ under f^{-1} is closed in $[0,1]$; i.e., the inverse image of any closed set in $[0,1]$ under f is closed. Therefore, f is continuous.