Practice Problem 5
Deadline: October 22, 2018

Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

Solution: First, notice that a square matrix M with integer entries has an inverse with integer entries iff $\det(M) = \pm 1$.

Now, let $f(x) = \det(A + xB)$. Thus $f(x)$ is a polynomial of degree at most 2 such that $f(x) = \pm 1$ for $x = 0, 1, 2, 3, 4$. So by pigeonhole principle f takes one of these values three or more times. But the only polynomials of degree at most 2 that take the same value three times are constant polynomials. In particular, $\det(A + 5B) = \pm 1$ and hence $A + 5B$ has an inverse with integer entries.