Practice Problem 6
Deadline: October 29, 2018

In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Solution: Player 0 wins with optimal play. In fact, we prove that Player 1 cannot prevent Player 0 from creating a row of all zeroes, a column of all zeroes, or a 2×2 submatrix of all zeroes. Each of these forces the determinant of the matrix to be zero.

For $i, j = 1, 2, 3$, let A_{ij} denote the position in row i and column j. Without loss of generality, we may assume that Player 1's first move is at A_{11}. Player 0 then plays at A_{22}:

$$\begin{bmatrix} 1 & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix}$$

After Player 1's second move, at least one of A_{23} and A_{32} remains vacant. Without loss of generality, assume A_{23} remains vacant; Player 0 then plays there. After Player 1's third move, Player 0 wins by playing at A_{21} if that position is unoccupied. So assume instead that Player 1 has played there. Thus of Player 1's three moves so far, two are at A_{11} and A_{21}. Hence for i equal to one of 1 or 3, and for j equal to one of 2 or 3, the following are both true:

a) The 2×2 submatrix formed by rows 2 and i and by columns 2 and 3 contains two zeroes and two empty positions.

b) Column j contains one zero and two empty positions.

Player 0 next plays at A_{ij}. To prevent a zero column, Player 1 must play in column j, upon which Player 0 completes the 2×2 submatrix in (a) for the win.