Practice Problem 7
Deadline: November 5, 2018

Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

Remember that $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

Solution: Note that for $m > n+1$, both binomial coefficients are nonzero and their ratio is

$$\frac{\binom{m}{n-1}}{\binom{m-1}{n}} = \frac{m!n!(m-n-1)!}{(m-1)!(n-1)!(m-n+1)!} = \frac{mn}{(m-n+1)(m-n)}.$$

Thus the condition $\binom{m}{n-1} > \binom{m-1}{n}$ is equivalent to $(m-n+1)(m-n) - mn < 0$. The left hand side of this last inequality is a quadratic equation of m with roots

$$\alpha(n) = \frac{3n-1+\sqrt{5n^2-2n+1}}{2}$$

and

$$\beta(n) = \frac{3n-1-\sqrt{5n^2-2n+1}}{2},$$

both of which are real since $5n^2 - 2n + 1 = 4n^2 + (n-1)^2 > 0$; it follows that m satisfies the given inequality if and only if $\beta(n) < m < \alpha(n)$. We conclude that $M(n)$ is the greatest integer strictly less than $\alpha(n)$, and thus that $\alpha(n) - 1 \leq M(n) < \alpha(n)$. Now

$$\lim_{n\to\infty} \frac{\alpha(n)}{n} = \lim_{n\to\infty} \frac{3 - \frac{1}{n} + \sqrt{5 - \frac{2}{n} + \frac{1}{n^2}}}{2} = \frac{3+\sqrt{5}}{2}$$

and similarly, $\lim_{n\to\infty} \dfrac{\alpha(n)-1}{n} = \dfrac{3+\sqrt{5}}{2}$. So by the sandwich theorem, we have

$$\lim_{n\to\infty} \frac{M(n)}{n} = \frac{3+\sqrt{5}}{2}.$$