

TRU Math Competition Practice Problem 7

Deadline: November 26, 2018

Given a nonempty set A , let $f : A \rightarrow A$ and $g : A \rightarrow A$ be two functions where

$$f(a) = g(f(f(a))) \text{ and } g(a) = f(g(f(a))),$$

for all $a \in A$. Prove that $f = g$.

Solution: Let $a \in A$. Then

$$f(a) = g(f(f(a))) = f(g(f(f(f(a)))))) = f(g \circ f^3(a)).$$

From $f(a) = g(f(f(a)))$ we have $f^2(a) = (f \circ f)(a) = f(g(f(f(a))))$.

So

$$\begin{aligned} f(a) &= f(g(f^3(a))) = f(g(f(f(f(a)))))) \\ &= f^2(f(a)) = f^2(g(f^2(a))) = f(f(g(f(f(a)))))) \\ &= f(g(f(a))) = g(a). \end{aligned}$$

Consequently, $f = g$.