

Practice Problem 8

Deadline: November 12, 2018

Let f be a three times differentiable function (defined on \mathbb{R} and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

Solution: Let $g(x) = e^{x/2}f(x)$. Then g has at least 5 distinct real zeros, and by repeated applications of Rolle's Theorem, g', g'', g''' have at least 4, 3, 2 distinct real zeros, respectively. But

$$g'''(x) = \frac{1}{8}e^{x/2}(f(x) + 6f'(x) + 12f''(x) + 8f'''(x))$$

and $e^{x/2}$ is never zero, so we obtain the desired result.