

Practice Problem 9

Deadline: November 19, 2018

Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Solution: Choose a set of a rows r_1, \dots, r_a considering an $a \times b$ submatrix whose entries are all 1. Then for $i, j \in \{1, \dots, a\}$, we have $r_i \cdot r_j = n$ if $i = j$ and 0 otherwise. Hence

$$\sum_{i,j=1}^a r_i \cdot r_j = an.$$

On the other hand, the term on the left is the dot product of $r_1 + \dots + r_a$ with itself; i.e., its squared length. Since this vector has a in each of its first b coordinates, the dot product is at least a^2b . Hence $an \geq a^2b$ and $n \geq ab$.