Practice Problem 10
Deadline: November 26, 2018

Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

Solution: Let $v_1, \cdots, v_n$ denote the rows of A. The determinant is unchanged if we replace v_n by $v_n - v_{n-1}$, and then v_{n-1} by $v_{n-1} - v_{n-2}$, and so forth, eventually replacing v_k by $v_k - v_{k-1}$ for $k \geq 2$. Since v_{k-1} and v_k agree in their first $k-1$ entries, and the k-th entry of $v_k - v_{k-1}$ is $\frac{1}{k} - \frac{1}{k-1}$, the result of these row operations is an upper triangular matrix with diagonal entries $1, \frac{1}{2} - 1, \frac{1}{3} - \frac{1}{2}, \cdots, \frac{1}{n} - \frac{1}{n-1}$. The determinant is then

$$\prod_{k=2}^{n} \left(\frac{1}{k} - \frac{1}{k-1} \right) = \prod_{k=2}^{n} \left(\frac{-1}{k(k-1)} \right) = \frac{(-1)^{n-1}}{(n-1)!n!}.$$