TRU Math Competition Practice Problem 9

Deadline: January 28, 2019

Let f be a twice differentiable function such that $f''(t) < 0$, for all $t \in \mathbb{R}$. Prove that if for two real numbers x and y we have

$$f'(y) + x < f(y+1),$$

then $f(y) > x$.

Solution by Benjamin Friedman: Let $L : \mathbb{R} \to \mathbb{R}$ be the linear approximation of f at y. So

$$L(t) = f'(y)(t-y) + f(y).$$

Then $L(y+1) = f'(y) + f(y)$. Since $f'' < 0$ everywhere, and $y+1 > y$, we see that

$$f(y+1) < L(y+1) = f'(y) + f(y).$$

We also know $f'(y) + x < f(y+1)$. Adding the two inequalities, we obtain

$$f(y+1) + f'(y) + x < f'(y) + f(y+1) + f(y) \Rightarrow x < f(y).$$