TRU Math Competition Practice Problem 10

Deadline: February 4, 2019

Let A be an $n \times n$ matrix such that, for every matrix B with trace zero, we have $tr(AB) = 0$. Prove that A is a diagonal matrix.

[*Hint: Consider the matrix* E_{ij} *whose* ij*-entry is* 1 *and zero elsewhere*.]

Solution by Benjamin Friedman: If $1 \leq i, j \leq n$ with $i \neq j$, then $tr(E_{ij}) = 0$. By the hypothesis, $a_{ij} = tr(AE_{ij}) = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$. Thus, A is a diagonal matrix.