TRU Math Competition Practice Problem 11

Deadline: February 11, 2019

Solve the recurrence relation $a_{n+2} = a_{n+1}a_n$, where $n \geq 0, a_0 = 1, a_1 = 2$.

Solution by Elizabeth McKenzie-Case: Let $a_{n+2} = a_{n+1}a_n$. Then

$a_0 = 1 = 2^0$, $a_1 = 2 = 2^1$, $a_2 = 2 \times 1 = 2^{1+0} = 2^1$, $a_3 = 2^1 \times 2^1 = 2^{1+1} = 2^2$, $a_4 = 2^2 \times 2^1 = 2^{2+1} = 2^3$, $a_5 = 2^3 \times 2^2 = 2^{2+3} = 2^5$, $a_6 = 2^5 \times 2^3 = 2^{5+3} = 2^8, \cdots$

So we have the following claim:

Claim. $a_n = 2^{F_n}$, where F_n is the n-th Fibonacci number.

Proof. We prove it by induction:

- Basic cases trivially hold.
- Assume $a_i = 2^{F_i}$ holds, for every $0 \leq i \leq k$. Then

$$a_{k+1} = a + ka_{k-1} = 2^{F_k}2^{F_{k-1}} = 2^{F_k+F_{k-1}} = 2^{F_{k+1}}.$$

Therefore, $a_n = 2^{F_n}$, where F_n is the n-th Fibonacci number.