TRU Math Competition Practice Problem 12

Deadline: February 18, 2019

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing, continuous function such that $(f \circ f \circ f)(x) = x$, for all $x \in \mathbb{R}$. Prove that $f(x) = x$, for all $x \in \mathbb{R}$.

Solution by Benjamin Friedman: Suppose there is an $x \in \mathbb{R}$ such that $f(x) > x$. Since f is increasing, we must have $f(f(x)) > f(x)$ and hence $f(f(f(x))) > f(f(x))$. Combining the last three inequalities, we obtain $f(f(f(x))) > x$, which is in contradiction with the assumption $f(f(f(x))) = x$, for every $x \in \mathbb{R}$. So $f(x) > x$ is not possible.

Similarly, we can show that $f(x) < x$ is impossible. Therefore, $f(x) = x$, for all $x \in \mathbb{R}$.