TRU Math Competition Practice
Problem 13
Deadline: February 25, 2019

Let A be an $n \times n$ matrix and I be the identity matrix. Show that $A = kI + B$, for some $k \in \mathbb{R}$ and an $n \times n$ matrix B where $tr(B) = 0$.

Solution by Benjamin Friedman: Since

$$A = \frac{tr(A)}{n} I + \left(A - \frac{tr(A)}{n} I \right),$$

we can set $k = \frac{tr(A)}{n}$ and $B = A - \frac{tr(A)}{n} I$. Note that

$$\begin{aligned} tr(B) &= tr\left(A - \tfrac{tr(A)}{n} I \right) \\ &= tr(A) - tr\left(\tfrac{tr(A)}{n} I \right) \\ &= tr(A) - \tfrac{tr(A)}{n} n \\ &= 0. \end{aligned}$$