TRU Math Competition Practice Problem 14

Deadline: March 4, 2019

Prove that for any integer $n \geq 2$, we have

$$2^n < \binom{2n}{n} < 4^n.$$

Solution by Benjamin Friedman: A **binary matrix** is a matrix such that each entry is 0 or 1.

Let C be the set of all $n \times 2$ binary matrices (n rows and 2 columns). There are 4 possibilities for each row: $(0\ 0), (0\ 1), (1\ 0)$, and $(1\ 1)$. Since there are n rows, we see that

$$|C| = 4^n.$$

Let B be the set of all $n \times 2$ binary matrices such that exactly n of the entries are 1's. Since each element of B corresponds to a selection of n entries out of $2n$, we see that

$$|B| = \binom{2n}{n}.$$

Let A be the set of all $n \times 2$ binary matrices such that each row contains a 1 and a 0. Thus there are two possibilities for each row: $(0\ 1)$ and $(1\ 0)$. Since there are n rows, we see that

$$|A| = 2^n.$$

The last step is to note that A is a proper subset of B, which is a proper subset of C. Thus $|A| < |B| < |C|$.