THOMPSON RIVERS UNIVERSITY | Mathematics and Statistics

TRU2019

1st Annual TRU Mathematics Competition

March 9, 2019
9:00-12:00

Student Name: ______________________________

Student Number: ____________________

No collaboration or outside aid is permitted. Please turn off and stow away your cell phones. All work is to be done on the problem papers or on the extra blank papers that are provided by the supervisor. All work to justify a solution and all necessary steps of a proof should be presented. Do not make reference to your work on different problems. No calculators are allowed.

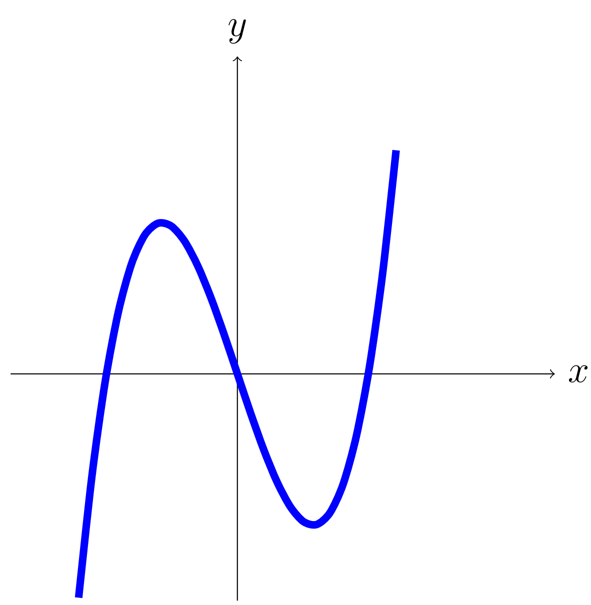
Place all the problems and extra papers you wish to have graded into the envelope in order of the problem numbers. Do not include scratch papers or blank papers in the envelope.

Each problem is graded on a basis of 0 to 10 points.

\Math{Club}

1. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

Solution: First, if we solve $x^4 + 36 \leq 13x^2$ for x, we get $x \in [-3, -2] \cup [2, 3]$. Since $f'(x) = 3x^2 - 3$, the only critical numbers of $f(x)$ are $x = \pm 1$, which is not in the interval $[-3, -2] \cup [2, 3]$. Considering the graph of $f(x)$



we conclude that f is increasing on $[-3, -2] \cup [2, 3]$ and hence the maximum value is $f(3) = 18$.

2. Let n be an odd positive integer. Prove that there are n consecutive positive integers whose sum is the square of an integer.

Solution: Since n is odd, $\frac{n-1}{2}$ is an integer. Now,

$$\left(\tfrac{n-1}{2}+1\right)+\left(\tfrac{n-1}{2}+2\right)+\left(\tfrac{n-1}{2}+3\right)+\cdots+\left(\tfrac{n-1}{2}+n\right)=$$
$$n\tfrac{n-1}{2}+(1+2+3+\cdots+n)=\tfrac{n(n-1)}{2}+\tfrac{n(n+1)}{2}=n^2.$$

3. Imagine you work at a farm and you don't know how many chickens are running around at the farm. However, you know that if you sell 75 chickens, then the chicken food will run out 20 days after the normal time. But if you buy 100 chickens, then the chicken food will run out 15 days before the normal time. Find the number of chickens.

Solution: Let x be the number of chickens, f the amount of food a chicken consumes in one day, and t the number of days before the food runs out. The total amount of food in the farm is

$$txf = (t+20)(x-75)f = (t-15)(x+100)f,$$

which can be written as two equations

$$\begin{cases} txf = (t+20)(x-75)f \\ (t+20)(x-75)f = (t-15)(x+100)f \end{cases}.$$

After some simplification, we get

$$\begin{cases} x = 5t \\ 20t - 3x - 300 = 0 \end{cases}$$

and hence $x = 300$.

4. Let f be a continuous and increasing function from $[a, b]$ into $[a, b]$ and $f(a) = a$. Prove that if $E = \{x \in [a, b] \mid x \leq f(x)\}$, then $f(E) = E$.

Solution: We first show that $E \subseteq f(E)$: Pick an arbitrary $x \in E$. So $x \leq f(x)$. If $x = f(x)$, we are done. If $x < f(x)$, then $f(a) = a \leq x < f(x)$ and hence, by the Intermediate Value Theorem, for some $c \in [a, x)$ we have $x = f(c)$ and therefore $x \in f([a, b])$. If $x \notin E$, then $f(c) < c$ and since f is increasing, we conclude that $f(x) = f(f(c)) < f(c) = x$, which is a contradiction. So $c \in E$ and hence $x \in f(E)$.

Now we show that $f(E) \subseteq E$: Pick an arbitrary $y \in f(E)$. So, for some $x \in E$, we have $y = f(x)$. Since $x \in E$, we have $x \leq f(x)$ and since f is increasing, we obtain $y = f(x) \leq f(f(x)) = f(y)$; i.e., $y \in E$.

5. If $f : [0,1] \to [0,1]$ is a continuous function, show that the equation

$$2x - \int_0^x f(t)dt = 1$$

has exactly one solution in $[0,1]$.

Solution: Define

$$g(x) = 2x - \int_0^x f(t)dt - 1.$$

Since f is continuous, g is differentiable on $(0,1)$. So $g'(x) = 2 - f(x)$. For all $x \in (0,1)$, we have $0 \leq f(x) \leq 1$, which implies $g'(x) = 2 - f(x) \geq 1 > 0$ and hence g is strictly increasing on $[0,1]$. Therefore, g has at most one zero on $[0,1]$.

Note that $g(0) = -1 < 0 < 1 - \int_0^1 f(t)dt = g(1)$. By the Intermediate Value Theorem, there is a $c \in [0,1]$ such that $g(c) = 0$; i.e., $2c - \int_0^c f(t)dt = 1$ for exactly one c in $[0,1]$.

6. Let $A = (a_{ij})$ be an $n \times n$ matrix such that, for every $1 \leq i \leq n$, we have $\sum_{j=1}^{n} a_{ij} = a$. If $A^2 = I$ (where I is the $n \times n$ identity matrix), find all possible values of a.

Solution: Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. We have

$$A\mathbf{v} = \begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = a\mathbf{v}.$$

Also,

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(a\mathbf{v}) = a(A\mathbf{v}) = a^2\mathbf{v}.$$

But $A^2 = I$. So $A^2\mathbf{v} = \mathbf{v}$. Therefore, $a^2\mathbf{v} = \mathbf{v}$ and hence $a = \pm 1$.

7. Evaluate the determinant of an $n \times n$ matrix whose diagonal entries are all equal to 0 and whose off-diagonal entries are all equal to 1.

Solution: If we subtract the first row from other rows, we obtain

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

Now, if we add the second column, then third column, and so on, to the first column, we obtain

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix} = \det \begin{bmatrix} n-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

which is as upper triangular matrix and its determinant is the product of its diagonal entries $(n-1)(-1)^{n-1}$.

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