William Lowell PUTNAM Mathematical Competition

Problem Set 1

Deadline: October 16, 2019

1. Prove that the expression

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$, where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $gcd(m,n)$ means the *greatest common divisor*.

2. Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Solution:

1. $gcd(m,n)$ can be written as an integer linear combination of m and n, that is, for some integers k,l, we have $km+ln = gcd(m,n)$. It follows that

$$\begin{aligned} \tfrac{gcd(m,n)}{n}\binom{n}{m} &= \tfrac{km+ln}{n}\binom{n}{m} \\ &= k\tfrac{m}{n}\binom{n}{m} + l\tfrac{n}{n}\binom{n}{m} \\ &= k\binom{n-1}{m-1} + l\binom{n}{m} \end{aligned}$$

which is an integer.

2. Replace x with $\frac{a}{x}$ to get

$$f'\left(\frac{a}{\frac{a}{x}}\right) = \frac{\frac{a}{x}}{f\left(\frac{a}{x}\right)} \Rightarrow f'(x) = \frac{a}{xf\left(\frac{a}{x}\right)}.$$

If we differentiate both sides, we obtain

$$f''(x) = \frac{-a}{x^2 f\left(\frac{a}{x}\right)} + \frac{a^2}{x^2 f(x) f^2\left(\frac{a}{x}\right)}.$$

Using $f'(x) = \frac{a}{xf\left(\frac{a}{x}\right)}$, we have

$$f''(x) = \frac{-f'(x)}{x} + \frac{(f')^2(x)}{f(x)} \Rightarrow xf(x)f''(x) = -f(x)f'(x) + x(f')^2(x).$$

If we divide both sides by $f^2(x)$, we obtain

$$\frac{xf''(x)}{f(x)} + \frac{f'(x)}{f(x)} - \frac{x(f')^2(x)}{f^2(x)} = 0,$$

which is the same as $\left(\frac{xf'(x)}{f(x)}\right)' = 0$. Therefore, $\frac{xf'(x)}{f(x)}$ is a constant c and integrating $\frac{f'(x)}{f(x)} = \frac{c}{x}$, we obtain $f(x) = dx^c$, for some $d > 0$.