Problem Set 2
Deadline: September 23, 2019

Let f be a twice-differentiable real-valued function satisfying

$$f(x)+f''(x)=-xg(x)f'(x),$$

where $g(x)\geq 0$, for all real x. Prove that $|f(x)|$ is bounded. [Hint. Multiply by $f'(x)$.]

Solution (Kateryna Tretiakova and Jonathan D. Gilchrist): If we multiply both sides of the given equality by $f'(x)$, we obtain

$$f(x)f'(x)+f'(x)f''(x)=-xg(x)(f'(x))^2.$$

The left side is $\frac{1}{2}(f^2(x)+(f')^2(x))'$. So

$$\frac{1}{2}(f^2(x)+(f')^2(x))'=-xg(x)(f'(x))^2.$$

If $x>0$, then the right side is negative and hence $f^2(x)+(f')^2(x)$ is decreasing. If $x<0$, then the right side is positive and $f^2(x)+(f')^2(x)$ is increasing. This means $f^2(x)+(f')^2(x)$ has a maximum at $x=0$ and hence it is bounded. This implies that $|f(x)|$ is also bounded.