



William Lowell
PUTNAM
Mathematical Competition

Problem Set 3

Deadline: September 30, 2019

A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function $g(x)$ defined on (a, b) such that this wrong product rule is true for x in (a, b) .

Solution: (Benjamin Friedman, Elizabeth McKenzie-Case, Kateryna Tretiakova, Jonathan Gilchrist, ...!)

We are looking for a solution to $f'g + fg' = f'g'$. Rearranging things, we have

$$\frac{g'}{g} = \frac{f'}{f' - f} = \frac{2xe^{x^2}}{(2x-1)e^{x^2}} = \frac{2x}{2x-1} = 1 + \frac{1}{2x-1}.$$

Integrating both sides, we obtain $\ln(g) = x + \ln|2x-1|^{1/2} + c$ and hence $g(x) = Ce^x \ln|2x-1|^{1/2}$.