William Lowell PUTNAM Mathematical Competition

Problem Set 5

Deadline: October 21, 2019

Prove that there exists a unique function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x) \text{ and } f(x) > 0 \; \forall x > 0.$$

Solution: Let $a_0 = x, a_a = f(x), a_2 = f(f(x)), a_3 = f(f(f(x))), \cdots$. Then the given functional equation implies that $a_{n+2} + a + n + 1 - 6a_n = 0$. The zeros of the characteristic polynomial $t^2 + t - 6$ are -3 and 2. So, for constants α and β, we have $a_n = \alpha(-3)^n + \beta(2)^2$. Since $f(x) > 0$, α must be zero, otherwise for some large values of n, we will have $a_n < 0$. Therefore, $a_n = \beta 2^n$. In particular, $f(x) = a_1 = 2a_0 = 2x$. So $f(x) = 2x$.