TRU Math Competition Practice Problem 1

Deadline: January 31, 2020

1. If n is a positive integer, prove that
$$\int_0^1 (\ln x)^n dx = (-1)^n n!.$$

   Solution: Start with integration by parts $u = (\ln x)^n, dv = dx$ and get $du = \frac{n(\ln x)^{n-1}}{x}, v = x$. So
$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx.$$

   Apply the integration by parts a few more times to get
$$\begin{aligned}\int (\ln x)^n dx &= x(\ln x)^n - nx(\ln x)^{n-1} + n(n-1)x(\ln x)^{n-2} \\ &+ \cdots + (-1)^{n-1} n(n-1)(n-2)\cdot\ldots\cdot 2x(\ln x) \\ &+ (-1)^n n(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1x.\end{aligned}$$

   Using L'Hospital's Rule repeatedly, we can see that $\lim_{t\to 0^+} x(\ln x)^{n-k} = 0$, for every $k = 0, \cdots, n-1$. Evaluating the following improper integral
$$\begin{aligned}\int_t^1 (\ln x)^n dx &= x(\ln x)^n]_t^1 - nx(\ln x)^{n-1}]_t^1 + n(n-1)x(\ln x)^{n-2}]_t^1 \\ &+ \cdots + (-1)^{n-1} n(n-1)(n-2)\cdot\ldots\cdot 2x(\ln x)]_t^1 \\ &+ (-1)^n n(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1x]_t^1\end{aligned}$$
   and sending t to zero finishes the proof.

2. Let f be a continuous function on $[0,1]$. Show that if $-1 \leq f(x) \leq 1$ for all $x \in [0,1]$, then there is $c \in [0,1]$ such that $[f(c)]^2 = c$.

Solution: Define $g(x) = f^2(x) - x$. It is a continuous function on $[0,1]$ and since $g(0) = f^2(0) \geq 0$ and $g(1) = f^2(1) - 1 \leq 0$, by the Intermediate Value Theorem there is a $c \in [0,1]$ such that $g(c) = 0$. That is, for some $c \in [0,1]$ we have $f^2(c) = c$.