TRU Math Competition Practice Problem 2

Deadline: February 7, 2020

1. Let a be a positive real number that is not an integer and let

$$n = \lfloor \frac{1}{a - \lfloor a \rfloor} \rfloor.$$

Prove that $\lfloor (n+1)a \rfloor - 1$ is divisible by $n+1$. [Hint: $\lfloor x \rfloor$ denotes the largest integer so that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$]. (University of Toronto, Undergraduate Mathematics Competition)

Solution: Since

$$n \le \frac{1}{a - \lfloor a \rfloor} < n+1,$$

and hence

$$n(a - \lfloor a \rfloor) \le 1 < (n+1)(a - \lfloor a \rfloor),$$

we have

$$1 + (n+1)\lfloor a \rfloor < (n+1)a \le 1 + n\lfloor a \rfloor + a < 2 + (n+1)\lfloor a \rfloor$$

and therefore $\lfloor (n+1)a \rfloor = 1 + (n+1)\lfloor a \rfloor$.

2. Let $f : [0,1] \to [0,1]$ be a surjective function such that

$$|f(x) - f(y)| \ge |x - y|,$$

for all $x, y \in [0,1]$. Show that f is continuous.

Solution: We first show that f is injective: Let $f(x_1) = f(x_2)$. Then $0 = |f(x_1) - f(x_2)| \geq |x_1 - x_2|$ implies that $x_1 = x_2$.

So, f is a bijection and hence it has an inverse. We show that f^{-1} is continuous at every point in $[0,1]$: For an arbitrary $\varepsilon > 0$, we define $\delta := \varepsilon$ and hence if $|x-y| < \delta$, then

$$|f^{-1}(x) - f^{-1}(y)| \leq |f(f^{-1}(x)) - f(f^{-1}(y))| = |x-y| < \delta = \varepsilon.$$

So, f^{-1} is continuous. Since $[0,1]$ is compact, f^{-1} is a closed map and hence the image of any closed set in $[0,1]$ under f^{-1} is closed in $[0,1]$; i.e., the inverse image of any closed set in $[0,1]$ under f is closed. Therefore, f is continuous.

3. Suppose that A and B are two square matrices of the same size for which the indicated inverses exist. Prove that

$$\left(A + AB^{-1}A\right)^{-1} + (A+B)^{-1} = A^{-1}.$$

Solution:

$$\begin{aligned}
\left(A + AB^{-1}A\right)^{-1} + (A+B)^{-1} &= \left(A(I + B^{-1}A)\right)^{-1} + (A+B)^{-1} \\
&= \left(AB^{-1}(B+A)\right)^{-1} + (A+B)^{-1} \\
&= (B+A)^{-1}BA^{-1} + (A+B)^{-1} \\
&= (A+B)^{-1}(BA^{-1} + I) \\
&= (A+B)^{-1}(B+A)A^{-1} \\
&= A^{-1}
\end{aligned}$$