

TRU Math Competition Practice Problem 3

Deadline: February 21, 2020

Let $f : [0, 1] \rightarrow \mathbb{R}$ be an integrable function.

Prove that

$$\int_0^1 \left(\int_x^1 f(t) dt \right) dx = \int_0^1 t f(t) dt.$$

Solution: Changing the order of integration, we can write:

$$\begin{aligned} \int_0^1 \left(\int_x^1 f(t) dt \right) dx &= \int_{x=0}^{x=1} \left(\int_{t=x}^{t=1} f(t) dt \right) dx \\ &= \int_{t=0}^{t=1} \left(\int_{x=0}^{x=t} f(t) dx \right) dt \\ &= \int_{t=0}^{t=1} f(t) \left(\int_{x=0}^{x=t} dx \right) dt \\ &= \int_0^1 t f(t) dt. \end{aligned}$$