TRU Math Competition Practice Problem 4

Deadline: February 28, 2020

1. Solve the recurrence relation $a_{n+2} = a_{n+1}a_n$, where $n \geq 0, a_0 = 1, a_1 = 2$.

   Solution: Let $a_{n+2} = a_{n+1}a_n$. Then $a_0 = 1 = 2^0$, $a_1 = 2 = 2^1$, $a_2 = 2 \times 1 = 2^{1+0} = 2^1$, $a_3 = 2^1 \times 2^1 = 2^{1+1} = 2^2$, $a_4 = 2^2 \times 2^1 = 2^{2+1} = 2^3$, $a_5 = 2^3 \times 2^2 = 2^{2+3} = 2^5$, $a_6 = 2^5 \times 2^3 = 2^{5+3} = 2^8, \cdots$ So we have the following claim: *Claim.* $a_n = 2^{F_n}$, where F_n is the n-th Fibonacci number. *Proof.* We prove it by induction:

   - Basic cases trivially hold.
   - Assume $a_i = 2^{F_i}$ holds, for every $0 \leq i \leq k$. Then

   $$a_{k+1} = a + ka_{k-1} = 2^{F_k}2^{F_{k-1}} = 2^{F_k+F_{k-1}} = 2^{F_{k+1}}.$$

   Therefore, $a_n = 2^{F_n}$, where F_n is the n-th Fibonacci number.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing, continuous function such that $(f \circ f \circ f)(x) = x$, for all $x \in \mathbb{R}$. Prove that $f(x) = x$, for all $x \in \mathbb{R}$.

   Solution: Suppose there is an $x \in \mathbb{R}$ such that $f(x) > x$. Since f is increasing, we must have $f(f(x)) > f(x)$ and hence $f(f(f(x))) > f(f(x))$. Combining the last three inequalities, we obtain $f(f(f(x))) > x$, which is in contradiction with the assumption $f(f(f(x))) = x$, for every $x \in \mathbb{R}$. So $f(x) > x$ is not possible. Similarly, we can show that $f(x) < x$ is impossible. Therefore, $f(x) = x$, for all $x \in \mathbb{R}$.

3. Let A be an $n \times n$ matrix and I be the identity matrix. Show that $A = kI + B$, for some $k \in \mathbb{R}$ and an $n \times n$ matrix B where $tr(B) = 0$.

   Solution: Since
$$A = \frac{tr(A)}{n} I + \left(A - \frac{tr(A)}{n} I \right),$$
   we can set $k = \frac{tr(A)}{n}$ and $B = A - \frac{tr(A)}{n} I$. Note that
$$\begin{aligned} tr(B) &= tr\left(A - \tfrac{tr(A)}{n} I\right) \\ &= tr(A) - tr\left(\tfrac{tr(A)}{n} I\right) \\ &= tr(A) - \tfrac{tr(A)}{n} tr(I) \\ &= 0. \end{aligned}$$