TRU II Math Battle, 2022

Without mathematics, there's nothing you can do.
Everything around you is mathematics.
Everything around you is numbers.

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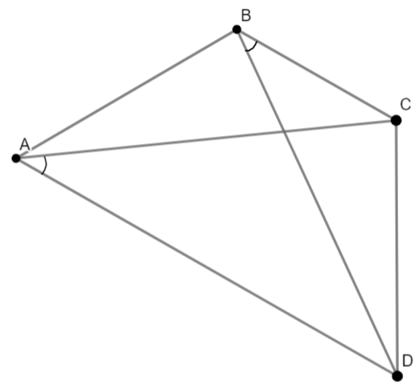

1. There are n non-zero numbers written on a board with $n > 2$. During each step you select two numbers a and b from the board, erase them and write down the numbers $a + \frac{b}{2}$ and $b - \frac{a}{2}$ instead. Can you generate the original set of numbers on the board after a finite number of non-zero steps?

2. Find all natural numbers k and n such that

$$k^2 + k^4 + k^6 = 10^n + 2022$$

3. In a quadrilateral $ABCD$, $\angle BAD = \angle CDA = 60°$ and $\angle CDB = \angle CAD$. Prove that $AB + CD = AD$.

4. The coefficients of the quadratic function $f(x) = ax^2 - bx + c$ are the powers of two (each of them is equal to 2^n for some real n). Prove that if the roots of the equation are integers, then the equation has only one root in real numbers (two repeated real roots).

5. The equilateral triangle with side lengths n is cut into equilateral triangles with sides of unit length. How many parallelograms are there inside of the given triangle that have sides which are parallel to the sides of the larger triangle?

6. Find all polynomials $p(x)$ with real coefficients such that for each real x we have

$$p(p(x)) = (x^2 + x + 1)p(x)$$

7. Let $q \neq 1$. Denote for arbitrary natural expressions

$$A_n = 1 + q + q^2 + \ldots + q^n$$

and

$$B_n = 1 + \frac{q+1}{2} + \frac{(q+1)^2}{2^2} + \ldots + \frac{(q+1)^n}{2^n}$$

Prove that

$$\binom{n+1}{1} + \binom{n+1}{2}A_1 + \binom{n+1}{3}A_2 + \ldots + \binom{n+1}{n+1}A_n = 2^n B_n$$

8. For which natural n there exists a non-regular polygon with all of its interior angles being equal?

Organized by TRUSU Math Club under K. Tretiakova